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Global Warming Potentials and multi-greenhouse gas price instruments

Abstract. The paper revisits the concept of Global Warming Potentials (GWP) from an economic perspective. Multi-greenhouse gas issues are analyzed using a general, dynamic, welfare-maximizing framework. The results confirm that the GWP induces a bias when used as the economic equivalence rule between greenhouse gases. In the linear damage case, we underline the trade-off between the time horizon used in GWP computation and the social discount rate. The standard definition of the GWP is shown to implicitly define different discount rates for different gases. A geometric interpretation of the second-best GWP-based emission tax is provided. It is shown that the change in the emission mix is such that total CO_2 equivalent emissions are the same under first-best and GWP-based emission tax regimes only if damages are linear. In the quadratic damage case, emission trajectories and steady-state solutions are compared under the first-best and GWP-based emission tax regimes.

Keywords: Global Warming Potential; Climate Change; price instruments.

JEL codes: Q25.

Introduction

The ratification of the Kyoto Protocol by Russia cleared the way for the adoption of the so-called 'Kyoto flexibility instruments'. On November 18, 2004, the UNFCCC[1] registered the first Clean Development Mechanism (CDM): *Brazil NovaGerar Landfill Gas to Energy* (EcoSecurities, Ltd, 2004).[2] Although carbon dioxide is the most scrutinized greenhouse gas (GHG) in the climate change debate, CO_2 emissions are only indirectly targeted in this very first project. More surprisingly, the project proposes to *emit* –rather than abate– CO_2. Indeed, the project entails collecting landfill methane to produce electricity. The key point in this project lies in the fact that methane's Global Warming Potential (GWP) is 21 times higher[3] than that of CO_2, while the combustion reaction transforms each ton of methane into only 2.75 tons of CO_2 (or 44/16, the molar mass ratio). Hence, just through the conversion from a higher- into a lower-GWP gas, total GHG emissions are reduced approximately eightfold (from 21 to 2.75) on a CO_2-*equivalent* basis. This example is far from being anecdotal; a number of projects currently under review by the UNFCCC reap advantage from the same kind of between-gases arbitrages.

The above example illustrates well the importance of non-CO_2 gases for the design of economic instruments. The multi-pollutant nature of climate change has prompted a fierce debate over the agreement architecture, especially with regard to the inclusion of non-CO_2 GHGs and the delineation of the 'Kyoto basket of gases' (Article 5.3 of the Kyoto Protocol). Empirical studies have clearly established that multi-gas mitigation strategies economically dominate CO_2-only strategies. The underlying intuition is easy to understand: multi-gas targets make it possible to take advantage of the most cost-effective abatement options, and thus lower the cost of achieving any given reduction. The magnitude of the estimated cost-savings, although varying with models and assumptions, unambiguously favors multi- over single-gas strategies (Hayhoe et al., 1999; Reilly et al., 1999).

The equivalence rule used in those studies for GHG comparison purposes is based on the GWP index. From the above example, one easily foresees the key role that this index plays in

[1] United Nation Framework Convention on Climate Change

[2] The interested reader is referred to `http://cdm.unfccc.int/Projects` for further information about this and other CDM projects.

[3] Using the 1995 IPCC Assessment Report's estimate as prescribed in the Kyoto Protocol. In its third Assessment Report, the IPCC has revised methane's GWP to 23.

setting 'relative prices' for greenhouse gases. Direct climate impact and atmospheric lifetime greatly vary from one gas to another. The GWP reflects the time-integrated radiative forcing resulting from one emission pulse of any GHG relative to that of CO_2 (Intergovernmental Panel on Climate Change, 2001). The Kyoto Protocol, which made its use mandatory in the reporting of States' emissions, gave the GWP a status with regard to international law that few other physics concepts can claim.

Is the GWP index the right indicator to compare greenhouse gases from an economic perspective? In short, the answer is no. The concept behind GWP raises a number of issues. A number of these issues are equally raised when dealing with any indicator that attempts to aggregate "apples and oranges" (OECD, 2002, replace here apples by carbon dioxide, and oranges by methane). Some authors have pointed out the simplified representation of the climate system the GWP relies on, questioning the relevance of this metric as an accurate climate change indicator (Smith and Wigley, 2000; Fuglestvedt et al., 2003; Godal, 2003). But the most fundamental criticisms are based on economic arguments. A small but growing amount of research has pointed out fundamental shortcomings in the GWP definition whenever this concept is used in economic assessments of multi-gas mitigation strategies. The lack of discounting, the overlooking of non-linearities in damage functions, and arbitrary time horizons are the most often raised criticisms (Reilly and Richards, 1993; Kandlikar, 1996; Bradford, 2001; Manne and Richels, 2001; Tol et al., 2003). Arguably, these shortcomings lead to distortions in the economic value of abatements in various greenhouse gases, in particular between short- and long-lived greenhouse gases. As a direct consequence, policies based on the GWP concept are accused of misleading the time path of resource allocation in abatement efforts.

Yet, since its introduction in the first IPCC assessment report in the early nineties (Lashof and Ahuja, 1990), the vast majority of assessments and emission reports that had to deal with multi-greenhouse gas issues have relied on the GWP. The GWP still stands as a key-concept in the toolbox of policy-makers and climate scientists alike. In fact, the GWP has proved both more effective and more operational as a negotiation basis than any alternative index found in the literature. One possible reason for this may be that, rightfully or not, it is easier to reach an agreement on an index summarizing the radiative forcings and atmospheric lifetimes, which are viewed by the Parties as well-documented and scientifically-sound, "hard" facts, than on an index that heavily depends on an economic measure of climate-change related damages and long-term discount rates.

The question addressed in this paper is: To what extent can GWPs be used in the design of climate change economic instruments? The paper thus investigates the analytical properties of GWP-based economic instruments. Two main routes have been followed in the economic literature in order to tackle the GWP issue (see for instance Delucchi and Lipman, 2003). The first approach consists in analyzing least-cost trajectories needed to meet a given target, typically expressed in terms of concentration (e.g. stabilization), aggregate radiative forcing, or temperature change ceiling. The cost-effectiveness approach has the advantage of ruling out major difficulties regarding the economic evaluation of climate damages and mitigation benefits. Total costs of achieving a given target can then be computed using alternative indexes, and the costs be compared against those associated with GWP (Manne and Richels, 2001; O'Neil, 2003; Shine et al., 2005; Sarofim et al., 2005). The second approach in contrast aims at solving the "Grand Problem" (Bradford, 2001) of deriving optimal paths that minimize the sum of abatement and climate damage costs (Reilly and Richards, 1993; Kandlikar, 1996; Moslener and Requate, 2001, for instance).

The model developed in this paper is more akin to the latter approach, as optimal trajectories of various GHGs are analytically investigated. However, it departs from previous literature in several respects. First, abatement costs are not explicitly modeled as in cost-minimizing models (e.g. Kandlikar, 1996; Moslener and Requate, 2001), but rather represented by the induced substitutions in consumption patterns and their impacts on (non-environmental) welfare. This more general formulation allows in particular to account for non-separable abatement cost functions. Second, we analyze the trade-off between the GWP's time horizon and the social discount rate. This discussion would not be possible in the framework retained by Kandlikar (1996) for instance, where the social planner's time horizon is assumed to be the same as the one used in the computation of the GWP. Third, the focus is less on alternative metrics to the GWP than on the properties of GWP-based economic instruments.

The paper is organized as follows. Section 1 reviews and discusses the main arguments supporting the critical views of the GWP. Section 2 presents the general analytical framework used to address multi-greenhouse gas issues in a dynamic, welfare-maximizing setting. The main features of the analytical general solution of the dynamic system are discussed. In section 3, the general expression of the optimal GWP-based emission tax is derived. In the particular case of a linear relationship between damage and concentrations, we underline the fundamental trade-off between the time horizon chosen in the computation of GWP and the discount rate (section 4). This analysis shows that an un-discounted measure based on a finite time horizon implicitly defines different discount rates for different gases. Section 5 presents the solution under a quadratic specification of the non-environmental welfare and constant emission factors. Section 6 examines the case of quadratic damage.

1. Global Warming Potentials: An economic perspective

When multi-gas targets are under consideration, they have to be formulated in a common unit, e.g., in tons of CO_2-equivalent. A metric is thus needed to compare GHGs. Fuglestvedt et al. (2003) provide an excellent and comprehensive survey of existing and alternative metrics of climate change. Among these metrics, the GWP is by far the most commonly used. This index was originally derived from a methodology developed for comparing ozone-depleting substances (Lashof and Ahuja, 1990).

It is worth having a closer look at the way this index is computed. Following the Intergovernmental Panel on Climate Change (2001), the definition of the concept writes as follows:

$$\mathrm{GWP}_{j,\mathrm{CO}_2}(\hat{T}) = \frac{\int_0^{\hat{T}} z_j^n(t)\theta_j(t)dt}{\int_0^{\hat{T}} z_{\mathrm{CO}_2}^n(t)\theta_{\mathrm{CO}_2}(t)dt} \tag{1}$$

where $z_j^n(t)$ and $z_{\mathrm{CO}_2}^n(t)$ represent the remaining atmospheric quantities of gas j and CO_2, respectively, at time t after an emission pulse of one mass unit at time $t = 0$. $\theta_j(t)$ and $\theta_{\mathrm{CO}_2}(t)$ represent the instantaneous radiative forcing of gas j and CO_2, respectively. The GWP thus represents the time-integrated radiative forcing of gas j relatively to that of CO_2 over a fixed and finite time horizon $\hat{T}$.

At first glance, the GWP concept might be regarded as a matter of interest for climate scientists and not for economists. As a matter of fact, this metric plays a crucial economic role, which is well illustrated in the example discussed in the introduction: it determines the relative 'prices' at which reductions in various GHGs can be traded.

In an insightful analysis of the debate over the relevance of the GWP index, O'Neil (2000) has identified two major difficulties. The first difficulty has to do with the selection of a criterion/variable in the long causality chain that proceeds from emissions to the economic costs of climate damages (see also Fuglestvedt et al., 2003). The GWP concept is centered around an early link in this chain (radiative forcing). In contrast, alternative indexes found in the economic literature (Reilly and Richards, 1993; Kandlikar, 1996; Manne and Richels, 2001) are based on the ratio of the marginal social values of concentration in each GHG. Hence, they are built up from the end of the causality chain (marginal abatement costs and/or marginal economic damages). It should therefore come as no surprise that indexes based on measures of different variables lead to diverging results as soon as the link between those variables does not reduce to a linear relationship.

One conclusion drawn from this discussion could be that conceptual difficulties arise less from the GWP definition itself than from its use as an economic equivalence rule. The GWP simply cannot accurately describe variables it was not meant to measure in the first place, unless very restrictive assumptions are made on the subsequent links in the causality chain identified by O'Neil. This is the main argument used by Kandlikar (1996), who highlights the implicit economic assumptions needed to have equivalence between the GWP and the optimal ratio of the respective shadow prices. These assumptions are a zero discount rate as well as linearity in the relationship between temperature change and economic damages. Kandlikar argues that these assumptions are hardly justified in the case of climate change. Again, this is just the reflect of the discrepancy between the very purpose of GWP, that is the equivalence at one particular link, and its use in welfare analyses.

The second difficulty lies in how the variable of interest should be measured. Are instantaneous measures appropriate? Should time-integrated measures be preferred? If the latter is chosen, what is the appropriate time horizon? Acknowledging the essentially dynamic nature of climate change, most indexes, such as Kandlikar's or the GWP, rely on time-integrated measures. In contrast to the GWP index, Kandlikar uses a discounted measure. Arguably, the use of an un-discounted and constant metric distorts the time-path allocation of abatements between short- and long-lived GHGs (see also Tol et al., 2003 and Michaelowa, 2003). As a result, un-discounted, constant metrics induce sub-optimal abatement paths.

To summarize, three main critical arguments are invoked against the use of GWP in economic analyses: (i) implicit assumption whereby radiative forcing and economic damage are linearly linked, (ii) absence of discounting, (iii) arbitrary time horizon. This text addresses these three criticisms in a stepwise manner. After the presentation of the general model (sections 2 and 3), we assume that (i) holds in order to focus on the last two criticisms (sections 4 and 5). We finally relax the linearity assumption in section 6.

2. Analytical framework: A multi-gas, optimal control problem

2.1. Formulation of the general problem

Consider an economy with a set of (private) goods ($i = 1, \ldots, m$). Equilibrium quantities at time t is denoted by the m-vector ${}^t\mathbf{x}_t = (x_{1t}, \ldots, x_{mt})$. Consumption and/or production causes emissions of a set of gases ($j = 1, \ldots, n$) in quantities ${}^t\boldsymbol{\varepsilon}(\mathbf{x}_t) = (\varepsilon_1(\mathbf{x}_t), \ldots, \varepsilon_n(\mathbf{x}_t))$. Greenhouse gases are stock pollutants. The change in atmospheric concentration in gas j (between current and pre-industrial levels) is denoted by z_{jt}, with ${}^t\mathbf{z}_t = (z_{1t}, \ldots, z_{nt})$. The equation of motion of z_j is given by the following equation:

$$\dot{z}_{jt} = \dot{z}_j^n(t) + \dot{z}_j^a(t) = -\tau_{jt} z_{jt} + \varepsilon_j(\mathbf{x}_t) \quad (j = 1, \ldots, n) \tag{2}$$

The two terms in equation (2) reflect the natural and anthropogenic components, respectively, in the accumulation of gas j in the atmosphere. Through natural absorption processes, z_{jt} decreases over time at rate τ_{jt}. In other words, if anthropogenic emissions are zero, atmospheric concentrations return to pre-industrial equilibrium levels at rate τ_{jt}. In general, τ_{jt} is not constant over time as it results from complex interactions between chemical species in the atmosphere, and might depend on the composition of the atmosphere itself. However, little is known about the functional form of τ_{jt}. In the rest of this paper, τ_j will be assumed constant. Average lifetime (or e-folding time) of gas j in the atmosphere is thus $1/\tau_j$.

The non-environmental part of welfare is measured through the function $U(\mathbf{x}_t)$. $U(.)$ is assumed to be continuously differentiable. It is assumed that there exists a unique vector $\mathbf{x}_t^{\mathrm{BAU}}$ (BAU as in *business-as-usual*) that maximizes $U(.)$. For all values of $\mathbf{x}$ that will be examined thereafter, we assume that $U(.)$ is increasing with respect to all x_i ($U'_{x_i} > 0$ for $x_i < x_i^{BAU}$).

The impact on climate is summarized through the change in average global temperature, denoted by $\theta(\mathbf{z})$. Generally speaking, the impact on average temperature depends on the concentrations of all gases. Each gas j contributes to the increase in temperature through its radiative forcing $\theta'_{z_j}(\mathbf{z})$, which is assumed to be positive[4].

Global economic damages are denoted by $D(\theta)$[5]. Following a commonly used assumption in the literature, $D(.)$ is assumed to be increasing and convex with respect to θ ($D'(.) > 0$ and $D''(.) \geq 0$). Climate change is assumed to be a 'pure' global externality: agents do not spontaneously internalize the effect on climate that their economic decisions induce. In addition, we will assume, that climate change causes an economic loss on agents that additively reduces total welfare. The social discount rate is denoted by δ. The problem faced by a (risk-neutral) social planner who intends to maximize total welfare over an infinite planning time horizon can be written as follows:

$$\max_{\mathbf{x}_t} \int_0^{\infty} [U(\mathbf{x}_t) - D(\theta(\mathbf{z}_t))]e^{-\delta t} dt \text{ subject to (2)} \tag{3}$$

Forming the (current) Hamiltonian of the problem, the optimality conditions are given by (the time index t is implicit and omitted):

$$\mathbf{x} \in \arg\max_{\mathbf{x}} \mathcal{H} = U(\mathbf{x}) - D(\theta(\mathbf{z})) - \sum_{j=1}^{n} \lambda_j(-\tau_j z_j + \varepsilon_j(\mathbf{x}_t)) \tag{4a}$$

$$\dot{\lambda}_j = \delta\lambda_j + \frac{\partial \mathcal{H}}{\partial z_j} \quad (j = 1, \ldots, n) \tag{4b}$$

Let $\mathbf{H}_{[U,\mathbf{x}]}$ and $\mathbf{H}_{[\varepsilon_j,\mathbf{x}]}$ be the $m \times m$-Hessian matrices of U and ε_j, respectively. The concavity of the Hamiltonian with respect to the command variable $\mathbf{x}$ is ensured by the fact that the Hessian matrix of $\mathcal{H}$ with respect to $\mathbf{x}$ is negative definite. Hence, we need the matrix $\left(\mathbf{H}_{[U,\mathbf{x}]} - \sum_{j=1}^{n} \lambda_j \mathbf{H}_{[\varepsilon_j,\mathbf{x}]}\right)$ to be non-singular and definite negative. All λ_j are non-negative at the optimum[6]. Therefore, standard assumptions on the concavity of $U(.)$ and

[4] Indeed, in a general approach, it would be possible to also deal with positive externalities, not only with negative ones. This would be the case, for instance, for carbon sequestration, which offsets some of the emissions and thus contributes to lower the pressure on climate. In this case, sequestered carbon can be accounted for as new 'gas' (negative emissions). The reasoning would remain unchanged, insofar as *business-as-usual* equilibrium quantities of carbon-sink enhancing goods should be provided in lower than optimal quantities.

[5] This representation relies on a simplified description of the climate system, whereby change in the radiative budget and average surface temperature are linearly linked through a constant climate sensitivity parameter (Intergovernmental Panel on Climate Change, 2001).

[6] Note that the expressions of the Hamiltonian (4a) and of the equations of motion of shadow prices (4b) are modified compared to their canonic expression in such a way that all λ_j are positive. In fact, the state

$\varepsilon_j(.)$ ($\mathbf{H}_{[U,\mathbf{x}]}$ negative definite and $\mathbf{H}_{[\varepsilon_j,\mathbf{x}]}$ positive semi-definite) are sufficient to ensure that the Hamiltonian is concave with respect to the command variables. The necessary conditions, whereby all derivatives of $\mathcal{H}$ with respect to the command variable are zero, are thus also sufficient to meet the static optimality conditions (4a):

$$\phi_i(\boldsymbol{\lambda},\mathbf{x}) = U'_{x_i}(\mathbf{x}) - \sum_{j=1}^{n} \lambda_j \varepsilon'_{j,x_i}(\mathbf{x}) = 0 \quad (i = 1, \ldots, m) \tag{5}$$

Equations (5) imply that, at each point in time, the marginal impact on welfare of any good i is equal to the sum of marginal emission content of x_i in all gases weighted by the respective shadow prices. At the optimum, λ_j thus reflects the marginal social value of emissions in gas j.

Equations (5) locally define a vector-valued function $\mathbf{f}(.)$ such that $\mathbf{x}^* = \mathbf{f}(\boldsymbol{\lambda})$ in the neighborhood of $\boldsymbol{\lambda}^*$, solution of (4a)–(4b). Hence, $\mathbf{f}(.)$ implicitly defines the optimal levels of consumption in all goods as a function of their social value. The conditions for using the implicit function theorem are met since the Jacobian matrix of $\boldsymbol{\phi}$ with respect to $\mathbf{x}$, $\mathbf{J}_{[\boldsymbol{\phi},\mathbf{x}]}(\boldsymbol{\lambda}) = \mathbf{H}_{[U,\mathbf{x}]} - \sum_{j=1}^{n} \lambda_j \mathbf{H}_{[\varepsilon_j,\mathbf{x}]}$, is non-singular in $\boldsymbol{\lambda}^*$ (see above). $\mathbf{J}_{[\mathbf{f},\boldsymbol{\lambda}]}$ represents the marginal change in consumption levels as a function of the shadow prices and is obtained through totally differentiating (5) with respect to $\mathbf{x}$ and $\boldsymbol{\lambda}$:

$$\mathbf{J}_{[\mathbf{f},\boldsymbol{\lambda}]}(\boldsymbol{\lambda}^*) = \left(\mathbf{H}_{[U,\mathbf{x}]}(\mathbf{x}^*) - \sum_{j=1}^{n} \lambda_j^* \mathbf{H}_{[\varepsilon_j,\mathbf{x}]}(\mathbf{x}^*) \right)^{-1} {}^t\mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}]}(\mathbf{x}^*) \tag{6}$$

The marginal effect of a change in the shadow prices is twofold: (i) consumed quantities $\mathbf{x}$ are adjusted in order to maximize $U(.)$ according to $\mathbf{H}_{[U,\mathbf{x}]}$; and (ii) marginal emission content of each good is modified as $\mathbf{x}$ changes, and these changes have also to be valued at the corresponding shadow prices. These two effects are embedded in the matrix in parenthesis. Note that the m-vector ${}^t\mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}]}(\mathbf{x}^*)\boldsymbol{\lambda}$ represents the marginal emission profile evaluated at prices $\boldsymbol{\lambda}$. The i-th entry of this vector is the total social value of emissions caused by good i at the margin.

$\mathbf{f}(\boldsymbol{\lambda})$ represents the social demand in private goods as a function of the 'prices' of greenhouse gases. Marginal abatement costs could thus be derived from equation (6) as the marginal loss in the non-environmental part of welfare resulting from a change in consumption levels.

variable here (concentrations) has a negative impact on the objective function. The canonic co-state variable attached to the equation of motion of z_j should thus also be negative. Without loss of generality, λ_j are chosen hereafter as the opposite of the standard shadow prices.

They have to be compared with the marginal social value of GHGs. Unlike cost-minimizing models (Kandlikar, 1996; Moslener and Requate, 2001), the formulation presented here is more general and, in particular, does not require any specific assumption about abatement costs separability. Note that $\mathbf{f}(\boldsymbol{\lambda})$ depends on the marginal substitution rates between all goods through $\mathbf{H}_{[U,\mathbf{x}]}$.

By replacing $\mathbf{x}$ by $f(\boldsymbol{\lambda})$ in equations (4b) and combining them with equations (2), the dynamics of shadow prices and concentrations can be rewritten as follows:

$$\dot{z}_j = -\tau_j z_j + \varepsilon_j(\mathbf{f}(\boldsymbol{\lambda})) \quad (j = 1, \ldots, n) \tag{7a}$$

$$\dot{\lambda}_j = (\delta + \tau_j)\lambda_j - D'(\theta(\mathbf{z}))\theta'_{z_j}(\mathbf{z}) \quad (j = 1, \ldots, n) \tag{7b}$$

2.2. Linearization and steady-state

Let $\boldsymbol{\Delta}_{[-\tau]}$ and $\boldsymbol{\Delta}_{[\delta+\tau]}$ be the $n \times n$-diagonal matrices with $(-\tau_j)$ and $(\delta + \tau_j)$ on the j-th component of the first diagonal, respectively. By taking the first-order Taylor's expansion of (7a)–(7b), the differential system is linearized in the neighborhood of any point $(\bar{\mathbf{z}}, \bar{\boldsymbol{\lambda}})$:

$$\begin{pmatrix} \dot{\mathbf{z}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{\Delta}_{[-\tau]} & \mathbf{A}(\bar{\boldsymbol{\lambda}}) \\ \mathbf{B}(\bar{\mathbf{z}}) & \boldsymbol{\Delta}_{[\delta+\tau]} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{pmatrix} \mathbf{a}(\bar{\boldsymbol{\lambda}}) \\ \mathbf{b}(\bar{\mathbf{z}}) \end{pmatrix} \tag{8}$$

where

$$\mathbf{A}(\bar{\boldsymbol{\lambda}}) = \mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}]}(\mathbf{f}(\bar{\boldsymbol{\lambda}}))\mathbf{J}_{[\mathbf{f},\boldsymbol{\lambda}]}(\bar{\boldsymbol{\lambda}}) \text{ and } \mathbf{a}(\bar{\boldsymbol{\lambda}}) = \boldsymbol{\varepsilon}(\mathbf{f}(\bar{\boldsymbol{\lambda}})) - \mathbf{A}(\bar{\boldsymbol{\lambda}})\bar{\boldsymbol{\lambda}} \tag{9a}$$

$$\mathbf{B}(\bar{\mathbf{z}}) = -D''.\mathbf{J}_{[\theta,\mathbf{z}]}(\bar{\mathbf{z}})\,{}^t\mathbf{J}_{[\theta,\mathbf{z}]}(\bar{\mathbf{z}}) - D'.\mathbf{H}_{[\theta,\mathbf{z}]}(\bar{\mathbf{z}}) \text{ and } \mathbf{b}(\bar{\mathbf{z}}) = -D'\mathbf{J}_{[\theta,\mathbf{z}]}(\bar{\mathbf{z}}) - \mathbf{B}(\bar{\mathbf{z}})\bar{\mathbf{z}} \tag{9b}$$

$\mathbf{A}(\bar{\boldsymbol{\lambda}})$ reflects the linear approximation of the optimal change in emissions resulting from a marginal change in the shadow prices, while $\mathbf{B}(\bar{\mathbf{z}})$ represents the change in damage caused by a marginal change in the concentrations. The steady state $(\mathbf{z}^\infty, \boldsymbol{\lambda}^\infty)$ of system (7a)–(7b) –if it exists– is defined by $\dot{\mathbf{z}} = 0$ and $\dot{\boldsymbol{\lambda}} = 0$. Generally speaking, it can be iteratively computed by solving the linearized system (8) in the neighborhood of $(\mathbf{z}^\infty, \boldsymbol{\lambda}^\infty)$.

2.3. Shadow price ratio dynamics

From the equation of motion for λ_j (7b), it can be easily seen that λ_j^* represents the integral over time of the flows of marginal damages caused by gas j, discounted by the discount rate (δ) and accounting for the absorption rate (τ_j). Hence, together with equation (5), this equation is the translation in a dynamic framework of the standard static result, whereby the marginal social value of emissions should be equal to the marginal damage. The ratio of the shadow prices of two arbitrarily chosen gases j and k thus sets the *relative* social value of gas j relatively to that of gas k. If gas k is chosen as the reference, the first-best tax on emissions in gas j that decentralizes the optimum is thus $\frac{\lambda^*_{jt}}{\lambda^*_{kt}}$ (gas k taken as the reference). In all generality, this ratio is not constant over time.

Differentiating the ratio with respect to time and using equation (7b) yields:

$$\dot{\left(\frac{\lambda_j}{\lambda_k}\right)} = \frac{\lambda_j}{\lambda_k}(\tau_j - \tau_k) + \frac{D'(\theta(\mathbf{z}))\theta'_{z_k}(\mathbf{z})}{\lambda_k}\left(\frac{\lambda_j}{\lambda_k} - \frac{\theta'_{z_j}(\mathbf{z})}{\theta'_{z_k}(\mathbf{z})}\right) \tag{10}$$

The evolution of the shadow price ratio over time is governed by two important quantities: (i) the difference in natural absorption rates $(\tau_j - \tau_k)$, and (ii) the ratio of average impact on temperature $(\theta'_{z_j}(\mathbf{z})/\theta'_{z_k}(\mathbf{z}))$. From equation (10), one also sees that if natural absorption rates are identical, the sign of the change in the shadow price ratio is only driven by the relative position of λ_j/λ_k with $\theta'_{z_j}(\mathbf{z})/\theta'_{z_k}(\mathbf{z})$.

Assume, without loss of generality, that $\tau_j \geq \tau_k$. That is, gas j is shorter-lived than gas k. Since λ_j and λ_k are positive, the first term in equation (10) is positive. Economic valuation of the damage is assumed to be increasing with respect to the average impact on temperature, and the marginal impact of concentration in gas k on temperature is positive. Therefore, the positivity of the second term in equation (10) depends on the relative position of shadow prices and radiative forcing ratios. If gas j, in addition to be shorter-lived than gas k, has a sufficiently lower direct impact on temperature –that is $\theta'_{z_j}(\mathbf{z})/\theta'_{z_k}(\mathbf{z})$ is sufficiently small– then λ_j/λ_k is increasing over time.[7] If radiative forcing of gas j is large enough relatively to that of gas k, the sign of (10) is ambiguous. This illustrates the possibility of a non-monotonic evolution of the shadow price ratio as found in Moslener and Requate (2001).

3. Multi-gas, GWP-based emission tax

In the problem described in (3), each gas is accounted for separately. Hence, in this case, there is no specific need to use a metric to compare or aggregate GHGs. Previous section

[7] Note that in the case of a constant radiative forcing ratio, if at any time λ_j/λ_k is greater than $\theta'_{z_j}(\mathbf{z})/\theta'_{z_k}(\mathbf{z})$, then λ_j/λ_k is also increasing over time.

has shown that the appropriate aggregator of the various GHGs is the n-vector of optimal shadow prices $\boldsymbol{\lambda}^*$. The components of $\boldsymbol{\lambda}^*$ are neither necessarily constant nor monotonic over time.

Assume now that a metric has been agreed upon and has been made mandatory so that the social planner cannot use it as a command variable. This metric converts emissions in any gas j into gas 1-equivalent. Let ${}^t\boldsymbol{\gamma} = (1, \gamma_2, \ldots, \gamma_n)$ be the n-vector of conversion coefficients of gas j into gas 1. All entries of $\boldsymbol{\gamma}$ are assumed to be constant over time[8]. Total emissions in tons of gas 1-equivalent are ${}^t\boldsymbol{\gamma} \cdot \boldsymbol{\varepsilon}(\mathbf{x}) = \sum_{j=1}^{n} \gamma_j \varepsilon_j(\mathbf{x})$. Let p_t be the emission tax expressed in dollars per ton of gas 1-equivalent.

The maximization of the non-environmental part of the welfare leads to the following conditions:

$$U'_{x_i}(\mathbf{x}_t) = p_t \sum_{j=1}^{n} \gamma_j \varepsilon'_{x_i,j}(\mathbf{x}) \quad (i = 1, \ldots, m) \tag{11}$$

Using an argument similar to the one used in section 2.1, equations (11) implicitly define the m-vector of consumption $\tilde{\mathbf{x}}_t$ as a vector-valued function $\mathbf{g}(p_t\boldsymbol{\gamma})$, which depends on the vector of emission tax on all gases converted into tons of gas 1-equivalent $(p_t\boldsymbol{\gamma})$. The problem faced by the social planner is thus modified as follows:

$$\max_{p_t} \quad \int_0^{\infty} \left[U(\mathbf{g}(p_t\boldsymbol{\gamma})) - D(\theta(\mathbf{z}))\right] e^{-\delta t} dt \tag{12a}$$

$$\text{s.t.} \quad \dot{z}_{jt} = -\tau_j z_{jt} + \varepsilon_j(\mathbf{g}(p_t\boldsymbol{\gamma})) \quad (j = 1, \ldots, n) \tag{12b}$$

The necessary conditions of optimality of the modified problem are:

$$p_t \in \arg\max_{p_t} \tilde{\mathcal{H}} = U(\mathbf{g}(p\boldsymbol{\gamma})) - D(\theta(\mathbf{z})) - \sum_{j=1}^{n} \mu_j(-\tau_j z_j + \varepsilon_j(\mathbf{g}(p\boldsymbol{\gamma}))) \tag{13a}$$

$$\dot{\mu}_j = \delta\mu_j + \frac{\partial\tilde{\mathcal{H}}}{\partial z_j} \quad (j = 1, \ldots, n) \tag{13b}$$

where ${}^t\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is the vector of the (modified) shadow prices attached to equations of motion of the state variable in problem (12a). Differentiating $\tilde{\mathcal{H}}$ with respect to p_t, using equation (11), and solving for p_t yields:

$$p_t = \frac{{}^t\boldsymbol{\gamma}\left(\mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}_t]}(\mathbf{g}(p_t\boldsymbol{\gamma}))\mathbf{J}_{[\mathbf{g},p_t\boldsymbol{\gamma}]}(p_t\boldsymbol{\gamma})\right)\boldsymbol{\mu}}{{}^t\boldsymbol{\gamma}\left(\mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}_t]}(\mathbf{g}(p_t\boldsymbol{\gamma}))\mathbf{J}_{[\mathbf{g},p_t\boldsymbol{\gamma}]}(p_t\boldsymbol{\gamma})\right)\boldsymbol{\gamma}} \tag{14}$$

[8] Observe that $\boldsymbol{\gamma}$ encompasses the standard definition of the GWP as a particular case, but also covers any kind of *constant* multi-GHG metric. As an illustration, CO_2-only strategies can also be analyzed using this framework (in this case, $\gamma_j = 0$ for all $j \geq 2$).

where $\mathbf{J}_{[\mathbf{g},p_t\boldsymbol{\gamma}]}(p_t\boldsymbol{\gamma})$ denotes the Jacobian matrix of $\mathbf{g}$.

Equation (14) gives the general expression of the emission tax based on the equivalence rule $\boldsymbol{\gamma}$. If for all j, γ_j is the standard GWP of gas j, then p_t is the optimal GWP-based emission tax. The optimal tax is locally defined as a linear combination of the optimal values of the shadow prices $\boldsymbol{\mu}$ in the modified problem. The j-th component of the n-vector ${}^t\boldsymbol{\gamma} \cdot \left(\mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}]} \cdot \mathbf{J}_{[\mathbf{g},p\boldsymbol{\gamma}]}\right)$ is the marginal change in emissions of gas j converted into gas 1-equivalent resulting from a change in the emission tax. The numerator in (14) is therefore equal to the marginal change in gas 1-equivalent emissions evaluated at current shadow prices $\boldsymbol{\mu}$. The denominator can be interpreted as a normalization factor. It also represents the marginal change in gas 1-equivalent emissions, but evaluated through the given equivalence rule $\boldsymbol{\gamma}$[9].

Equation (14) –when combined together with the definition of the Jacobian matrix of $\mathbf{g}(.)$, $\mathbf{J}_{[\mathbf{g},p_t\boldsymbol{\gamma}]}$– provides an interesting geometric interpretation of the optimal $\boldsymbol{\gamma}$-based emission tax. If ${}^t\mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}_t]}$ has full column rank –that is, the emission profile in any gas j (the j-th column of ${}^t\mathbf{J}_{[\boldsymbol{\varepsilon},\mathbf{x}_t]}$) cannot be obtained as a linear combination of the other gases' emission profiles– then the matrix appearing both at the numerator and the denominator in equation (14) is definite positive. Therefore, this matrix defines a norm $||.||$ in $I\!R^n$. Let $\cos(.,.)$ denote the cosine of the angle between two vectors in $I\!R^n$ defined according to the bilinear form associated to this norm. Then the optimal tax can be expressed as follows:

$$p_t = \frac{||\boldsymbol{\mu}_t||}{||\boldsymbol{\gamma}||}.\cos\left(\boldsymbol{\gamma},\boldsymbol{\mu}_t\right) \tag{15}$$

The optimal $\boldsymbol{\gamma}$-based emission tax can be decomposed into a scaling factor $(\frac{||\boldsymbol{\mu}_t||}{||\boldsymbol{\gamma}||})$, and a measure of the angle between $\boldsymbol{\gamma}$ and $\boldsymbol{\mu}_t$. The first factor is an aggregate measure of how much of the total social value of emissions is actually captured in $\boldsymbol{\gamma}$. The second factor summarizes the bias induced by $\boldsymbol{\gamma}$, i.e. how (in-)accurately $\boldsymbol{\gamma}$ reflects the relative social value of all individual gases.

The computation of $||\boldsymbol{\mu}_t||$, $||\boldsymbol{\gamma}||$, and $\cos\left(\boldsymbol{\gamma},\boldsymbol{\mu}_t\right)$ in equation (15) relies on a norm that weights the components of $\boldsymbol{\mu}_t$ and $\boldsymbol{\gamma}$ according to the marginal impact on welfare of changes in $\mathbf{x}_t$. In this general framework, this norm is defined locally[10]. Therefore, two effects drive

[9] The fact that there appears a normalization factor in the expression of p_t should come as no surprise. Indeed, γ is defined as a relative equivalence rule. That means that it should always be possible to change the reference gas (from carbon dioxide to methane for instance) without changing the optimal solution of the program (13a)–(13b). Obviously, this implies a corresponding change in the value of p_t.

[10] Further assumptions made in sections 5 and 6 restrict the analysis to the case where the weights used in $||.||$ are constant with respect to consumption levels.

the changes in the optimal $\boldsymbol{\gamma}$-based emission tax over time: *(i)* marginal impact on welfare are modified as $\mathbf{x}_t$ changes, which imply changes in the weights defining $||.||$, and *(ii)* changes in $\boldsymbol{\mu}_t$ the social value of individual GHGs.

Using $\mathbf{g}(p\gamma)$, the differential system can be rewritten as follows:

$$\dot{z}_j = -\tau_j z_j + \varepsilon_j(\mathbf{g}(p\gamma)) \quad (j = 1, \dots, n) \tag{16a}$$

$$\dot{\mu}_j = (\delta + \tau_j)\mu_j - D'(\theta(\mathbf{z}))\theta'_{z_j}(\mathbf{z}) \quad (j = 1, \dots, n) \tag{16b}$$

Following the same reasoning as above, the modified differential system is linearized in the neighborhood of any point $(\bar{\mathbf{z}}, \bar{\boldsymbol{\mu}})$ and written in matrix form:

$$\begin{pmatrix} \dot{\mathbf{z}} \\ \dot{\boldsymbol{\mu}} \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{\Delta}_{[-\tau]} & \frac{\tilde{\mathbf{A}}(\bar{\boldsymbol{\mu}})\gamma \; {}^t\gamma\tilde{\mathbf{A}}(\bar{\boldsymbol{\mu}})}{{}^t\gamma\tilde{\mathbf{A}}(\bar{\boldsymbol{\mu}})\gamma} \\ \mathbf{B}(\bar{\mathbf{z}}) & \boldsymbol{\Delta}_{[\delta+\tau]} \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\mu} \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{a}}(\bar{\boldsymbol{\mu}}) \\ \mathbf{b}(\bar{\mathbf{z}}) \end{pmatrix} \tag{17}$$

where $\tilde{\mathbf{A}}(\bar{\boldsymbol{\mu}})$ and $\tilde{\mathbf{a}}(\bar{\boldsymbol{\mu}})$ are similar to the definitions given in (9a), in which $\mathbf{J}_{[\mathbf{g},p\boldsymbol{\gamma}]}$ replaces $\mathbf{J}_{[\mathbf{f},\boldsymbol{\lambda}]}$. At this level of generality, it is difficult to be more conclusive about the dynamics of the system without taking more specific assumptions about the functions.

4. Linear damage and linear temperature change

Consider first the case of linear economic damage and linear change in temperature with respect to concentrations. These assumptions, although highly unrealistic in the case of climate change, have the advantage of simplifying the comparison between the GWP and the optimal shadow price ratio. Indeed, these assumptions rule out the first major difficulty discussed in section 1. This enables to focus on the two remaining shortcomings of the GWP, namely the arbitrary in the choice of the time horizon and the lack of discounting. To illustrate this, the following assumptions are made:

$$D(\theta) = \alpha.\theta \text{ with } \alpha > 0 \tag{H1}$$

$$\theta(\mathbf{z}) = {}^t\boldsymbol{\theta} \cdot \mathbf{z} \text{ with } \boldsymbol{\theta} = (\theta_j) \in I\!R^{+n} \tag{H2}$$

Under assumptions (H1) and (H2), equation (7b) reduces to a first-order linear differential equation with a constant, positive right-hand side. Solving this equation for gas j and gas k

and using the corresponding transversality conditions, the optimal shadow price ratio is:

$$\frac{\lambda_j^*}{\lambda_k^*} = \frac{\alpha\theta_j}{\delta+\tau_j} / \frac{\alpha\theta_k}{\delta+\tau_k} = \frac{\theta_j(\delta+\tau_k)}{\theta_k(\delta+\tau_j)} \tag{18}$$

Very important is the fact λ_j^*/λ_k^* is constant over time under (H1) and (H2). Because of the assumed linearity of the link between concentrations and damage, the social value of gas j relative to gas k does not change over time. Introducing (H2) in equation (1), the GWP –also constant over time by definition– writes:

$$\mathrm{GWP}_{j,k}(\hat{T}) = \frac{\theta_j\tau_k(1-e^{-\tau_j\hat{T}})}{\theta_k\tau_j(1-e^{-\tau_k\hat{T}})} \tag{19}$$

From equation (19), it is easily seen that the GWP differs from the ratio of optimal shadow prices for at least two reasons: (i) the chosen time horizon ($\hat{T}$) which appears in the expression of the GWP, but not in the shadow price ratio, and (ii) the discount rate (δ) that only affects the shadow price ratio.

Moreover, the sign of the difference between the GWP and the ratio of optimal shadow prices only depends on the natural decay rates (τ_j and τ_k), the discount rate (δ) and the time horizon used in the computation of the GWP ($\hat{T}$). Under (H1)-(H2), it does not depend on the marginal damage (α). Nor does it depend on the radiative forcing ratio ($\frac{\theta_j}{\theta_k}$).

PROPOSITION 1. *Consider any 2-uple of gases j and k such that $\tau_j \neq \tau_k$ and $\tau_j, \tau_k > 0$. Under (H1) and (H2) and for a given time horizon $\hat{T} > 0$:*

i) There exists a unique positive discount rate $\hat{\delta}_{j,k}(\hat{T})$ such that $GWP_{j,k}(\hat{T}) = \lambda_j^*/\lambda_k^*$.

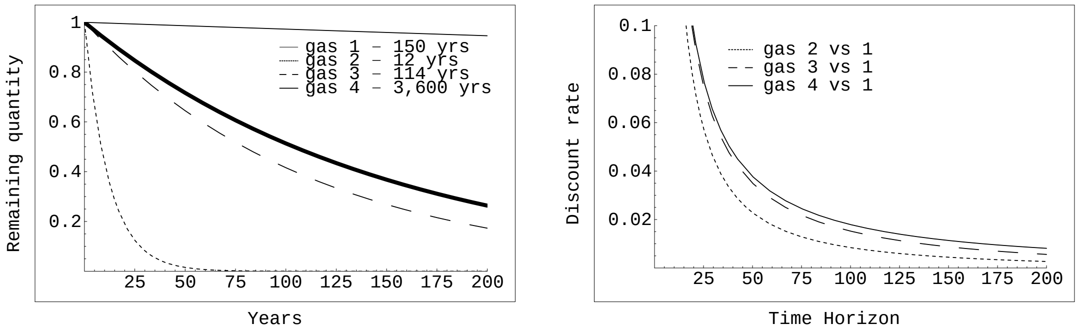
ii) If $\tau_j > \tau_k$, then $GWP_{j,k}(\hat{T}) < \lambda_j^*/\lambda_k^* \iff \delta > \hat{\delta}_{j,k}(\hat{T})$.

If $\tau_j < \tau_k$, then $GWP_{j,k}(\hat{T}) < \lambda_j^*/\lambda_k^* \iff \delta < \hat{\delta}_{j,k}(\hat{T})$.

Proof. See appendix

Proposition 1 gives interesting insights into the difference between GWP- and welfare-based indexes. It shows that this difference depends heavily on the discount rate and on the time horizon used in the computation of the GWP index. The proposition also illustrates the trade-off between those two quantities.

The results of proposition 1 are illustrated on Figure 1. Figure 1.a shows the natural evolution of concentrations in four gases. Gas 2, 3 and 4 are characterized by average lifetimes of 12



a. Evolution of concentrations for four illustrative constant decay rates

b. Combinations of $(\hat{T}, \delta)$ for which $\text{GWP}_{j,1}(\hat{T}) = \frac{\lambda_j^*}{\lambda_1^*}$

Figure 1. Trade-off between discount rate and time horizon (linear damage and linear temperature response)

(dotted), 114 (dashed) and 3,600 (solid) years, respectively. These lifetimes can be associated with CH_4, N_2O, and SF_6 respectively (Intergovernmental Panel on Climate Change, 2001). Gas 1 is assumed to have an average lifetime of 150 years (thick). This lifetime lies in the range given for CO_2[11]. For illustrative purposes, the chosen range of atmospheric lifetimes is deliberately very wide. As an illustration, the remaining fraction after 200 years of a unitary emission pulse of gas 2 is almost unnoticeable (order of magnitude 10^{-8}), whereas it is approximately 0.95 for gas 4.

Figure 1.b illustrates the trade-off between the time horizon $\hat{T}$ and the social discount rate. Gas 1 is taken as the reference. If $\hat{T}$ is fixed, for example at 100 years (the commonly used convention in the IPCC), then part (i) of the proposition indicates that there exists a unique value of the discount rate such that $\text{GWP}_{2,1}(100) = \frac{\lambda_2^*}{\lambda_1^*}$. For the shortest-lived gas, $\hat{\delta}_{2,1}(100)$ would thus be approximately 0.8%. For a 500-year time horizon, it drops to 0.02%. As for gas 4 –the longest-lived gas in our example– the characteristic values of the discount rate are higher (1.79% and 0.26% for 100 and 500 years, respectively). Since gas 4 is longer-lived than gas 2, a higher discount rate is needed to ensure the equality between the GWP and the shadow price ratio. It also follows that longer time horizons imply lower characteristic values of the discount rate. As the curves on Figure 1.b do not cross, one sees that there exists no 2-uple $(\delta, \hat{T})$ such that *all* GWPs could be equal to the respective shadow price ratio as soon as $n > 2$.

[11] Indeed, the behavior of CO_2 in the atmosphere is much more complex and cannot be satisfactorily summarized through a constant decay rate, as the 'speed' of exchanges between different carbon pools (terrestrial sinks, upper oceans, deep oceans, atmospheric) varies widely. More sophisticated approaches approximate the dynamics of carbon as a sum of exponential. This simplification, however, does not affect the general conclusions drawn in this section.

Conversely, one could set the discount rate and derive the time horizon that ensures equality between the GWP and the respective shadow price ratio. For commonly used values of the discount rates in long-term environmental issues, for example lower than 3%, the time horizons such that the GWP is equal to the optimal shadow price ratio appear reasonably close to the 100-year mark. For instance, for a discount rate of 1.5%, the time horizon that ensures equality between the shadow price ratio and the GWP is 67.4, 101.37, and 117.3 years (gas 2, 3, and 4, respectively). However, the computation of the GWP is quite sensitive to the value of the time horizon $\hat{T}$. Computing the GWP of gas 2 (resp. 4) over a time horizon of 67.4 (resp 117.3) years instead of 100 years indeed yields a 34% (resp. 5%) increase in the index. In addition, for very low discount rates sometimes advocated in the case of climate change, the difference in 'equivalent' time horizons widens between short- and long-lived GHG.

It is often argued that the GWP index over-estimates the social value of shorter-lived gases and undervalues longer-lived gases (O'Neil, 2003). Part (ii) of proposition 1 re-examines the validity of this assertion and highlights the crucial role that δ and $\hat{T}$ play in this statement. For a given time horizon $\hat{T}$, the GWP index overvalues (undervalues) gases that are shorter-lived (longer-lived) than the reference gas if the discount rate is small. The reverse statement holds if the discount rate is large. Results for a 100-year time horizon are shown on Table I. The first four rows correspond to the case examined in Figure 1 ($\tau_1 = 1/150$). Ranges of discount rate values for which the GWP is higher (lower) than the shadow-price ratio are indicated with a '+' ('-'). From Table I, one sees that the assertion whereby the GWP index overvalues shorter-lived gases only holds if the discount rate is small. For larger (but still reasonable) discount rates, the GWP overvalues shorter-lived gases and undervalues longer-lived gases. The last eight rows of Table I examine the sensitivity of this result with respect to two alternative assumptions regarding the decay rate of the reference gas.

What does this have to say about the validity of the GWP as a climate change index? First, this confirms the fact that the GWP is not an adequate metric whenever it is used to measure the *economic* equivalence between gases. This result is in line with findings from Kandlikar (1996)[12]. The linear case examined here greatly simplifies the link between radiative forcing and economic damage. This supposedly reduces the bias between impact-based and welfare-

[12] In his paper, Kandlikar assumes that the time horizon of the social planner's program is the same as the one used in the computation of the GWP. This restriction is not made in the present paper.

Table I. Discount rate ranges for which the GWP over- (+) or under-estimates (-) the relative social value of greenhouse gases (linear damage, linear temperature change, $\hat{T} = 100$)

Gas	Decay rate (τ_j)	Value of the discount rate δ			
		$]0, \hat{\delta}_{2,1}(100)[$	$]\hat{\delta}_{2,1}(100), \hat{\delta}_{3,1}(100)[$	$]\hat{\delta}_{3,1}(100), \hat{\delta}_{4,1}(100)[$	$]\hat{\delta}_{4,1}(100), +\infty[$
	$\tau_1 = 1/150$	$]0, 0.84\%[$	$]0.84\%, 1.52\%[$	$]1.52\%, 1.79\%[$	$[1.79\%, +\infty[$
Gas 2	1/12	+	-	-	-
Gas 3	1/114	+	+	-	-
Gas 4	1/3600	-	-	-	+
	$\tau_1 = 1/50$	$]0, 0.43\%[$	$]0.43\%, 1.20\%[$	$]1.20\%, 1.51\%[$	$[1.51\%, +\infty[$
Gas 2	1/12	+	-	-	-
Gas 3	1/114	-	-	+	+
Gas 4	1/3600	-	-	-	+
	$\tau_1 = 1/200$	$]0, 0.91\%[$	$]0.91\%, 1.57\%[$	$]1.57\%, 1.84\%[$	$[1.84\%, +\infty[$
Gas 2	1/12	+	-	-	-
Gas 3	1/114	+	+	-	-
Gas 4	1/3600	-	-	-	+

based indexes. Notwithstanding, a bias remains, rooting in the lack of discounting and the arbitrariness of the chosen time horizon.

Second, the results highlight the possible trade-off between the discount rate and the time horizon used in the GWP computation. One could argue that the choice of the discount rate is not more arbitrary than the choice of $\hat{T}$. The value of the discount rate has prompted fierce debates among economists, especially for long-term environmental issues such as climate change. The choice of any value for δ is *inter alia* contingent to assumptions about future preferences and productivity of capital. From Figure 1, the 100-year assumption might appear as a compromise, which implies reasonable implicit values of the discount rate. However, simple computations show that the GWP is quite sensitive to the time horizon, in particular for short-lived gases.

Third, the sign of the difference between the GWP of any gas (relative to a reference gas) and the respective shadow price ratio depends on the discount rate, the chosen time horizon and the difference in decay rates. Therefore for a given time horizon and a given discount rate, the GWP index undervalues some gases and overvalues others. Table I shows that, in the linear case, small changes in the social discount rate may revert the sign of these differences.

5. Linear damage, constant emission factors, and quadratic welfare

In order to further discuss the evolution over time of the system under (H1) and (H2), more specific assumptions are needed regarding the analytical form of emission factors and welfare. In this section, the following assumptions are thus made in addition to (H1) and (H2).

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{E} \cdot \mathbf{x} \text{ with } \mathbf{E} = (e_{ji}) \in {I\!\!R^{+}}^{n \times m} \tag{H3}$$

$$U(\mathbf{x}) = {}^{t}\mathbf{u}\mathbf{x} - \frac{1}{2}\,{}^{t}\mathbf{x}\mathbf{V}\mathbf{x} \text{ with } \mathbf{u} \in {I\!\!R^{+}}^{m},\ \mathbf{V} \in I\!\!R^{m \times m} \text{ symmetric positive definite} \tag{H4}$$

Assumption (H3) simply corresponds to a linear relationship between equilibrium quantities and emissions, and thus pertains to constant (and positive) emission factors. This assumption is in line with the use of constant emission factors as prescribed in IPCC inventory reports. (H4) is a stronger assumption. It corresponds to a linear-quadratic specification of the welfare function. Such a specification for $U(.)$ involves a linear demand system. In order to meet the concavity properties discussed in section 2.1 as well as standard properties of the demand system (symmetry, non-increasing demand functions), $\mathbf{V}$ is a symmetric positive definite $m \times m$-matrix.

Business-as-usual equilibrium quantities are such that they maximize $U(\mathbf{x})$. Under (H4), these quantities are $\mathbf{x}^{\text{BAU}} = \mathbf{V}^{-1}\mathbf{u}$. For the sake of simplicity, preferences are assumed to be constant over time. As a direct consequence, business-as-usual equilibrium quantities are therefore constant over time. Likewise, under (H3) associated emissions –given by $\boldsymbol{\varepsilon}^{\text{BAU}} = \mathbf{E}\mathbf{x}^{\text{BAU}} = \left(\mathbf{E}\mathbf{V}^{-1}\right)\mathbf{u}$– are also constant over time. The equation of motion of z_j thus reduces to a first-order linear differential equation with a constant and positive right-hand side. The right-hand side is defined by total optimal emissions in gas j, denoted by $\varepsilon_j^{\text{BAU}}$. The business-as-usual time path of z_j is therefore fully characterized by initial concentrations $\mathbf{z}_0$:

$$z_{jt}^{\text{BAU}} = (z_{j0} - \frac{\varepsilon_j^{\text{BAU}}}{\tau_j})e^{-\tau_j t} + \frac{\varepsilon_j^{\text{BAU}}}{\tau_j} \tag{20}$$

Under (H3) and (H4), business-as-usual concentration of gas j tends to $\varepsilon_j^{\text{BAU}}/\tau_j$ as t tends to infinity. z_j^{BAU} is monotonically decreasing (resp. increasing) over time if initial concentration z_{j0} is greater than (resp. lower than) steady-state concentrations.

Consider now the evolution of $\mathbf{z}$ if damages are taken into account. Optimal emissions $\boldsymbol{\varepsilon}^*$ are computed by introducing (H3) and (H4) in the optimality conditions (5), which yields:

$$\boldsymbol{\varepsilon}^* = \mathbf{E}\mathbf{x}^* = \left(\mathbf{E}\mathbf{V}^{-1}\right)\mathbf{u} - \left(\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\boldsymbol{\lambda}^* \tag{21}$$

(H3) and (H4) imply linear demand functions. Optimal equilibrium quantities $\mathbf{x}^*$ thus depend linearly on $\boldsymbol{\lambda}$. It is easy to verify that this is also the case for the reduction in consumption relative to the business-as-usual situation ($\mathbf{x}^{\mathrm{BAU}} - \mathbf{x}^* = \left(\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\boldsymbol{\lambda}^*$) and the optimal abatement ($\boldsymbol{\varepsilon}^{\mathrm{BAU}} - \boldsymbol{\varepsilon}^* = \left(\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\boldsymbol{\lambda}^*$).

As shown in section 4, $\boldsymbol{\lambda}^*$ is constant over time if (H1) and (H2) hold and λ_j^* is equal to $\alpha\theta_j/(\delta+\tau_j)$. This in turn implies constant emissions under (H4). Therefore, under assumptions (H1)-(H4), the equation of motion of z_j reduces to a first-order linear differential equation with a constant and positive right-hand side. The right-hand side is defined by total optimal emissions in gas j, denoted by ε_j^*. The solutions of this equation therefore take a form similar to (20), in which $\varepsilon_j^{\mathrm{BAU}}$ is replaced by ε_j^*. Therefore, the 'first-best' time-path of concentration in gas j (z_j^*) is monotonically decreasing (resp. increasing) if initial concentrations are greater than (resp. lower than) steady-state concentrations ε_j^*/τ_j.

Consider now the $\boldsymbol{\gamma}$-based emission tax. Introducing (H3) and (H4) into equation (14) yields:

$$p_t = \frac{{}^t\boldsymbol{\gamma}\left(\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\tilde{\boldsymbol{\mu}}}{{}^t\boldsymbol{\gamma}\left(\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\boldsymbol{\gamma}} \tag{22}$$

The $\boldsymbol{\gamma}$-based emission tax is a linear combination of the shadow prices μ_j. The coefficients in this linear combination depend on the conversion coefficients, the marginal substitution rates between all goods, and the emission factors. If rank($\mathbf{E}$)$= n$, the geometric interpretation provided in the general case by equation (15) still holds. This interpretation is however simpler, since the definition of norm $||.||$ is constant with respect to consumption levels under (H3)-(H4).

If (H1)-(H4) hold, emissions under a $\boldsymbol{\gamma}$-based emission tax regime ($\tilde{\boldsymbol{\varepsilon}}$) are:

$$\tilde{\boldsymbol{\varepsilon}} = \mathbf{E}\tilde{\mathbf{x}} = \mathbf{E}\mathbf{V}^{-1}\mathbf{u} - \frac{{}^t\boldsymbol{\gamma}\left(\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\tilde{\boldsymbol{\mu}}}{{}^t\boldsymbol{\gamma}\left(\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\boldsymbol{\gamma}}\left(\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\right)\boldsymbol{\gamma} \tag{23}$$

Under (H1) and (H2), the differential system characterizing $\tilde{\boldsymbol{\mu}}$ is the exactly same as the one characterizing $\boldsymbol{\lambda}^*$. The solutions are thus identical. Therefore, we have $\tilde{\boldsymbol{\mu}} = \boldsymbol{\lambda}^*$. As a direct consequence, emissions under a $\boldsymbol{\gamma}$-based emission tax regime are constant over time and the time-path of concentrations takes a form similar to the one described by equation (20). The results of this section are summarized in the following proposition.

PROPOSITION 2. *Under (H1)-(H4), the following results hold for any $\boldsymbol{\gamma} > 0$:*

i) ${}^t\boldsymbol{\lambda}^*(\boldsymbol{\varepsilon}^{BAU} - \boldsymbol{\varepsilon}^*) \geq 0$ *(>0 if $rank(\mathbf{E}) = n$) and* ${}^t\boldsymbol{\lambda}^*(\boldsymbol{\varepsilon}^{BAU} - \tilde{\boldsymbol{\varepsilon}}) \geq 0.$

ii) ${}^t\boldsymbol{\lambda}^*(\tilde{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}^*) \geq 0$

iii) ${}^t\boldsymbol{\gamma}(\boldsymbol{\varepsilon}^* - \tilde{\boldsymbol{\varepsilon}}) = 0$

Proof. See appendix

Proposition 2 focuses on *aggregated* emissions. How should emissions under BAU, 'first-best' and '$\boldsymbol{\gamma}$-based emission tax' regimes be compared? Two natural candidates are competing as aggregator. The first one is the vector of optimal shadow prices $\boldsymbol{\lambda}^*$. Emissions in all gas are weighted by the respective shadow prices. The total social value of emissions under the three regimes can thus be compared. Part (i) of proposition 2 indicates that total emissions under 'first-best' and '$\boldsymbol{\gamma}$-based emission tax' regimes are lower than business-as-usual emissions when aggregated according to the respective social value of each GHG. This should come as no surprise as climate change related damages are ignored in the business-as-usual scenario, whereas they are accounted for under the two alternative regimes. Both regimes thus induce an environmental amelioration compared to the *laissez-faire* situation. Part (ii) proposition 2 indicates that the total value of first-best emissions ${}^t\boldsymbol{\lambda}^* \cdot \boldsymbol{\varepsilon}^*$ is lower than the total value of emissions under any $\boldsymbol{\gamma}$-based emission tax regime ${}^t\boldsymbol{\lambda}^* \cdot \tilde{\boldsymbol{\varepsilon}}$. In other words, the $\boldsymbol{\gamma}$-based emission tax induces a net social loss.

Part (iii) of the proposition is more surprising. The second natural candidate as an aggregator is the $\boldsymbol{\gamma}$-equivalence rule. Result (iii) indicates that total emissions –if aggregated according to $\boldsymbol{\gamma}$– are the same under the 'first-best' and '$\boldsymbol{\gamma}$-based emission tax' regimes. This result is strongly related to the linear damage assumption, as it requires $\boldsymbol{\lambda}^* = \tilde{\boldsymbol{\mu}}$. It holds regardless of the choice of (non-negative) $\boldsymbol{\gamma}$. Stated differently, a GWP-based emission tax leads to the same total emissions expressed in *CO_2-equivalent* as in the first-best situation. Geometrically, this means that $\tilde{\boldsymbol{\varepsilon}}$ belongs to the hyperplane of $\mathbb{R}^n$ defined by ${}^t\boldsymbol{\gamma} \cdot \boldsymbol{\varepsilon} = {}^t\boldsymbol{\gamma} \cdot \boldsymbol{\varepsilon}^*$.

The $\boldsymbol{\gamma}$-based emission tax regime involves changes in emissions in individual gases compared to the first-best situation, but these changes are compensated when aggregated according to $\boldsymbol{\gamma}$. Emissions in some gases necessarily increase, while emissions in other gases decrease (expressed in tons of gases).

This result is graphically illustrated in the case $n = 2$ (Figure 2). Consider that the solution of program (3) leads to first-best emissions $\boldsymbol{\varepsilon}^* = (\varepsilon_1^*, \varepsilon_2^*)$. Total emissions in gas 1-equivalent are ${}^t\boldsymbol{\gamma} \cdot \boldsymbol{\varepsilon}^*$. All points $(\varepsilon_1, \varepsilon_2)$ along the line CD (slope $-\frac{\gamma_2}{\gamma_1}$) are characterized by the same total emissions in tons of gas 1-equivalent. Imagine that the optimal shadow price ratio $\frac{\lambda_2^*}{\lambda_1^*}$ is smaller than $\frac{\gamma_2}{\gamma_1}$. That is, the equivalence rule overvalues gas 2 relatively to gas 1. From (ii), we know that emissions under $\boldsymbol{\gamma}$-based regime $\tilde{\boldsymbol{\varepsilon}} = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)$ are necessarily above the line EF (slope $-\frac{\lambda_2^*}{\lambda_1^*}$). (iii) imposes that $\tilde{\boldsymbol{\varepsilon}}$ lies somewhere on the line CD. Therefore, $\tilde{\boldsymbol{\varepsilon}}$ thus lies on the segment AC. It follows that $\tilde{\varepsilon}_1$ is greater than ε_1^* and $\tilde{\varepsilon}_2$ is lower than ε_2^*. Consequently abatement in the overvalued GHG (here gas 2) is greater than optimal, whereas abatement in undervalued GHG (gas 1) is lower than optimal. One can apply the same reasoning to the case $\frac{\lambda_2^{*\prime}}{\lambda_1^{*\prime}} > \frac{\gamma_2}{\gamma_1}$.

Consider that $\boldsymbol{\gamma}$ is the vector of the standard GWPs. In this case, whether B or B' situation prevails depends on the value of the discount rate relative to $\hat{\delta}_{2,1}(\hat{T})$ and on the decay rate of gas 2 compared to that of gas 1 (see proposition 1).

6. Quadratic damage, constant emission factors, and quadratic welfare

6.1. First-best regime

In this section, the linearity assumption regarding the relationship between concentration and climate-change related damage is relaxed. Assumptions (H2)-(H4) are kept, while (H1) is replaced by (H1'):

$$D(\theta) = \frac{1}{2}\beta\theta^2 \text{ with } \beta > 0 \tag{H1'}$$

Under (H1'), the marginal damage is increasing with respect to θ. Therefore, marginal damage cannot be reduced to a proportional transformation of radiative forcing. $D'(\theta)$ depends on θ, which in turn depends on the levels of concentrations.

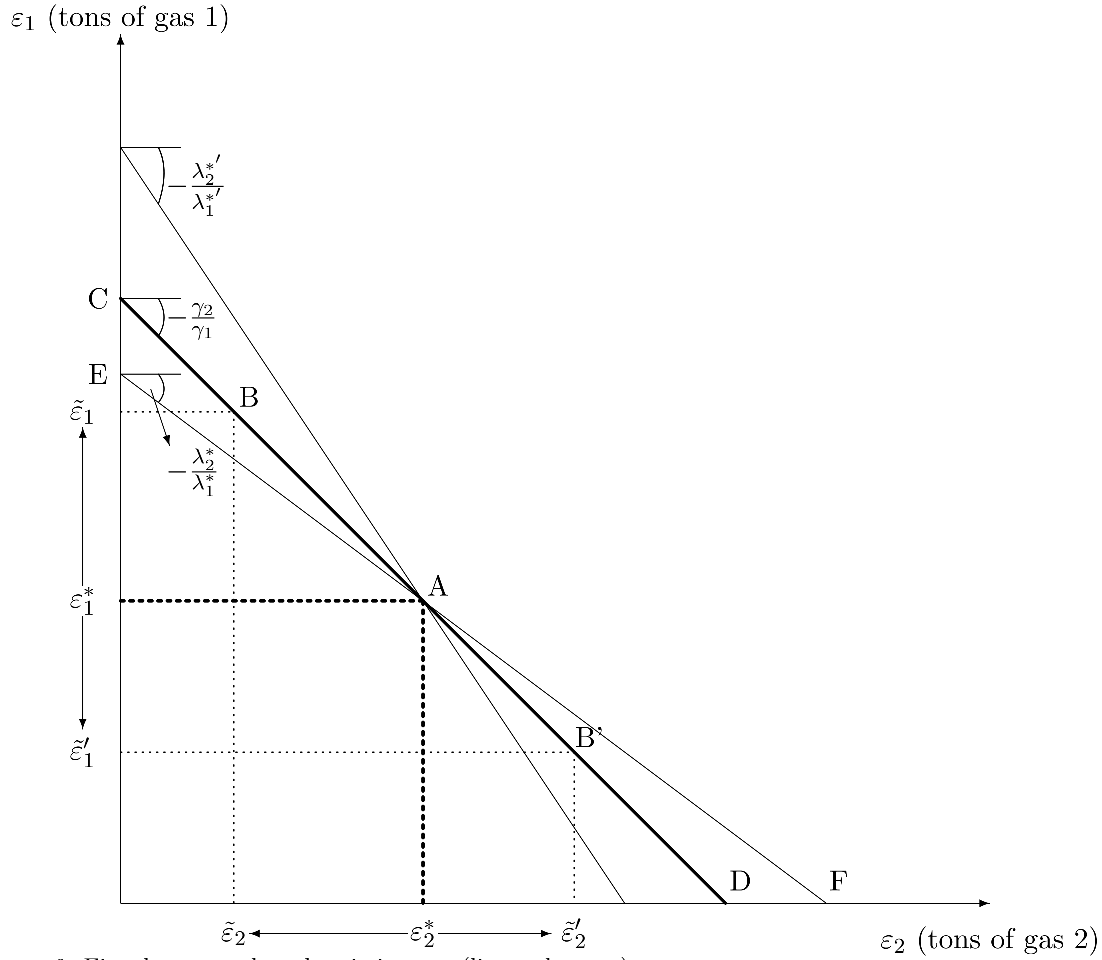


Figure 2. First-best vs $\boldsymbol{\gamma}$-based emission tax (linear damage)

Introducing (H1') and (H2)-(H4) into the linearized differential system (8) yields:

$$\begin{pmatrix} \dot{\mathbf{z}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Delta}_{[-\tau]} & -\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E} \\ -\beta\boldsymbol{\theta}\ {}^t\boldsymbol{\theta} & \boldsymbol{\Delta}_{[\delta+\tau]} \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{pmatrix} \mathbf{E}.\mathbf{V}^{-1}\mathbf{u} \\ \mathbf{0} \end{pmatrix} \tag{24}$$

Note first that the differential system is linear under (H1'), (H2)-(H4). Note also that (H1') and (H2) imply that $\mathbf{b}(\bar{\mathbf{z}})$ is equal to the null n-vector. Finally, remark that, as in previous section, $\mathbf{a}(\bar{\boldsymbol{\lambda}})$ is constant over time and equal to business-as-usual emissions.

Let ν_k $(k = 1, \ldots, 2n)$ be the eigenvalues of the $2n \times 2n$-matrix appearing in system (24) and $\boldsymbol{\Omega}$ the $2n \times 2n$-matrix of the $2n$ eigenvectors in columns $\boldsymbol{\omega}_k$. The general solution of system (24) is:

$$z_{jt}^* = \sum_{k=1}^{2n} \omega_{j,k}(c_k e^{-\nu_k t} + \frac{d_k}{\nu_k}) \text{ for all } j = 1, \ldots, n \tag{25a}$$

$$\lambda_{jt}^* = \sum_{k=1}^{2n} \omega_{n+j,k}(c_k e^{-\nu_k t} + \frac{d_k}{\nu_k}) \text{ for all } j = 1, \ldots, n \tag{25b}$$

c_k are $2n$ integration constants to be derived from initial concentrations, $\mathbf{z}_0$, and transversality conditions. d_k is the k-th entry of the $2n$-vector $\mathbf{d}$ defined as $\boldsymbol{\Omega}^{-1}\begin{pmatrix}\mathbf{E}\mathbf{V}^{-1}\mathbf{u}\\ \mathbf{0}\end{pmatrix}$.

In order to exhibit the analytical solution of system (24), we need to diagonalize a $2n\times 2n$-matrix. The behavior of the general solution of the system depends on the non-singularity of this matrix (existence of the steady state) and on the sign of the eigenvalues (saddle-point).

6.2. $\boldsymbol{\gamma}$-based emission tax regime

The same reasoning can be applied to the $\boldsymbol{\gamma}$-based emission tax regime:

$$\begin{pmatrix}\dot{\mathbf{z}}\\ \dot{\boldsymbol{\mu}}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Delta}_{[-\tau]} & -\dfrac{\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\boldsymbol{\gamma}\ {}^t\boldsymbol{\gamma}\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}}{{}^t\boldsymbol{\gamma}\mathbf{E}\mathbf{V}^{-1}\ {}^t\mathbf{E}\boldsymbol{\gamma}}\\ -\beta\boldsymbol{\theta}\ {}^t\boldsymbol{\theta} & \boldsymbol{\Delta}_{[\delta+\tau]}\end{pmatrix}\begin{pmatrix}\mathbf{z}\\ \boldsymbol{\mu}\end{pmatrix} + \begin{pmatrix}\mathbf{E}.\mathbf{V}^{-1}\mathbf{u}\\ \mathbf{0}\end{pmatrix} \tag{26}$$

Again, the existence of a steady state and the dynamic behavior of the general solution depend on the singularity of the matrix appearing in system (26) and the sign of the eigenvalues.

The comparison of the general solution under first-best and $\boldsymbol{\gamma}$-based emission tax regimes is not as straightforward as it was in the linear case. In particular, the result whereby $\boldsymbol{\gamma}$-aggregated emissions are the same under both regimes does not generally hold as soon as the damage is not linear with respect to concentrations. In other words, the bias induced by the use of the $\boldsymbol{\gamma}$ equivalence rule is twofold: (i) the optimal emission mix is changed compared to first-best regime; (ii) the change in the emission mix modifies the social value of individual greenhouse gases in such a way that total $\boldsymbol{\gamma}$-equivalent emissions also differ from their first-best levels.

6.3. Illustration with two gases

We focus on the two-gas case in order to analytically illustrate the properties of the dynamic system under the first-best and $\boldsymbol{\gamma}$-based emission tax regimes.

PROPOSITION 3 (Steady state). *In the case of two gases and under (H1'), (H2)-(H4):*

i) Both system (24) and system (26) admit a unique steady-state.

ii) Under both regimes, the steady-state is a saddle point.

Proof. See appendix.

Proposition 3 shows the existence and unicity of the steady state under both regimes. Moreover, in the case of two gases, the steady-state is shown to be a saddle-point. That is, two of the eigenvalues characterizing the 4×4 matrix appearing in system (24) (respectively (26)) are negative, whereas the two others are positive. Let ν_1 and ν_2 (respectively $\tilde{\nu_1}$ and $\tilde{\nu_2}$) be the positive eigenvalues, and ν_3 and ν_4 (respectively $\tilde{\nu_3}$ and $\tilde{\nu_4}$) the negative ones. c_3 and c_4 (respectively $\tilde{c}_3$ and $\tilde{c}_4$) –which are the coefficients associated with the negative eigenvalues in (24) (respectively (26))– are zero along the converging optimal path under the first-best regime (respectively the $\boldsymbol{\gamma}$-based emission tax regime).

PROPOSITION 4 (Steady-state shadow price ratio). *In the case of two gases ($\tau_2 > \tau_1$), under (H1'), (H2)-(H4):*

i) The shadow price ratio, taken at the steady-state, is such that: $\frac{\lambda_2^{*\infty}}{\lambda_1^{*\infty}} = \frac{\tilde{\mu}_2^{\infty}}{\tilde{\mu}_1^{\infty}} = \frac{\theta_2(\delta+\tau_1)}{\theta_1(\delta+\tau_2)}$.

ii) A sufficient condition for $\boldsymbol{\lambda}^{*\infty} \geq \tilde{\boldsymbol{\mu}}^{\infty}$ *to hold is* $\frac{\theta_2\tau_1}{\theta_1\tau_2} \leq \frac{\gamma_2}{\gamma_1} \leq \frac{\theta_2(\delta+\tau_1)}{\theta_1(\delta+\tau_2)}$ *(in the case $\tau_2 < \tau_1$, revert upper and lower bounds). The condition is also necessary if rank(**E**)= n.*

iii) A sufficient condition for $\begin{pmatrix} \mathbf{z}^{*\infty} \\ \boldsymbol{\lambda}^{*\infty} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{z}}^{\infty} \\ \tilde{\boldsymbol{\mu}}^{\infty} \end{pmatrix}$ *to hold is* $\frac{\gamma_2}{\gamma_1} = \frac{\theta_2(\delta+\tau_1)}{\theta_1(\delta+\tau_2)}$.

Proof. See appendix.

The steady-state shadow price ratios are the same under the first-best regime and a $\boldsymbol{\gamma}$-based emission tax regime (i). This result holds regardless of the choice of $\boldsymbol{\gamma}$. The optimal time-path and the steady-state shadow price of *individual* gases may differ. Nevertheless, the shadow-price ratio is preserved in the steady-state. Consequently, the discussion conducted in section 4 can be applied to the case of quadratic damage. Depending on the respective decay rates, the value of the discount rate and the time horizon chosen in the computation of the GWP, the GWP under- or overstates the steady-state social value of gas 2 relatively to gas 1.

Part (ii) of the proposition indicates when individual shadow prices are higher under first-best regime than under $\boldsymbol{\gamma}$-based emission tax regime. For this to hold, the conversion coefficient $\frac{\gamma_2}{\gamma_1}$ has to lie between the steady-state shadow prices ratio $(\frac{\theta_2(\delta+\tau_1)}{\theta_1(\delta+\tau_2)})$ and the infinite time-horizon GWP $(\frac{\theta_2\tau_1}{\theta_1\tau_2})$. The former represents the time-integrated relative marginal damage over the infinite planning horizon. The latter is the time-integrated relative impact on temperature. Stated differently, this means that, if the equivalence coefficient between gas 2 and gas 1 is neither too large nor too small, the steady-state social value of emissions will be greater in the first-best regime than under the γ-based emission tax regime.

Part (iii) of proposition 4 shows that well-chosen equivalence coefficient can actually lead to exactly the same steady-state concentrations and individual shadow prices. Again, this coefficient is $\frac{\theta_2(\delta+\tau_1)}{\theta_1(\delta+\tau_2)}$, the time-integrated relative marginal damage over the infinite planning horizon. In other words, if the GWP is used as the equivalence factor, then the time horizon used in the GWP computation should be chosen in such a way that it is consistent with the social discount rate (see discussion in section 4).

Concluding remarks

To the question: "What would be the optimal equivalence rule between various greenhouse gases, which differ in their atmospheric lifetime and impact on climate?", the economic answer is a welfare-based index that balances marginal economic damages, marginal abatement costs, and differentiates between short- and long-term impacts, for instance through discounting long-term damages. Propositions of this nature can be found in the economic literature, and the purpose of this paper was not to produce a new one.

The main point made in this paper is that economists were simply not asked this question. For various reasons, the IPCC assessment reports have promoted the use of the GWP index in multi-greenhouse gas assessments despite well-established shortcomings. The concept has been as successful as being included in the Kyoto Protocol itself. Economists thus have to deal with the concept, which will most likely remain as a key-feature in the on-going climate change negotiations. This paper proposes a general formulation of a GWP-based tax. This tax takes the equivalence rule as given –be it the GWP or any constant alternative metric. The general geometric interpretation of this tax explicitly highlights the bias induced by the use of a constant metric. The bias is proportional to the angle between the equivalence coefficients of all gases into a reference gas and the respective shadow prices. The measure of this bias depends on the marginal impacts on welfare of foregone consumption in all private goods.

The results of this paper thus confirm that GWPs do not provide an accurate measure of the relative economic value of GHGs. In this sense, the results corroborate those obtained from previous cost-effectiveness analyses. However, there is a major difference between the welfare-maximizing framework developed in the present paper and the cost-effectiveness approach. The latter examines least-cost mitigation strategies needed to meet a given concentration or temperature target. Consequently, the social value of abating short-lived GHGs –such as methane– is found to be very low. In short, the conclusion is that methane abatements should be used as a short-term "brake", only when approaching the target. Consequently, methane is found to be overvalued a great deal if attached a price 21 times higher than CO_2. We find that the question of whether the GWP over- or undervalues short-lived GHGs is indeed more complex when analyzed in a welfare-maximizing framework. The answer to this question depends heavily on the social discount rate, the time horizon used in the computation of the GWP, and the respective lifetime of the various gases. In particular, abatements in short-lived GHGs may actually be more desirable than it is reflected in the GWP if the social discount rate is sufficiently high.

How does the GWP-based emission tax regime compares to the first-best regime? If climate change related damages are linear with respect to concentrations, the abatement mix is modified because of the bias induced by the use of a constant metric (higher abatements in overvalued GHGs, lower abatements in undervalued GHGs). However, total emissions in CO_2-equivalent are preserved. This does not hold in the more realistic case of convex damage. In the latter case, the steady-state relative social value is shown to be unchanged. In general, however, steady-state values of concentrations as well as optimal trajectories are modified under a GWP-based emission tax regime, relatively to the first-best regime.

The GWP-emission tax can be thought as a second-best instrument. It yields higher welfare values than the "naive" interpretation of the GWP, whereby non-CO_2 GHGs prices are directly obtained as the marginal value of CO_2 emissions times the respective GWP. The optimal distribution of the abatement burden among sectors and/or countries strongly depends on how multi-gas issues will be dealt with and on the design multi-gas instruments in further developments of climate policy. As an illustration, the optimal levels of abatement that need to be achieved in agriculture –the major emitting sector for non-CO_2 gases– is highly sensitive to the relative prices assigned to methane and nitrous oxide. Policy and economic implications of the use of the GWP-based emission tax are therefore potentially large.

This work should be extended in several directions. We only mention two directions for further research. First, the properties of the GWP-based tax have to be further investigated from an empirical perspective, using realistic economic and climate parameters. Second, the sensitivity of the system to uncertainties in key-parameters, such as the marginal damage or the temperature response to concentrations is still to be tested. The economic intuition is that the GWP-based emission tax is less sensitive to changes in the value of marginal damage. This can be tested using the dynamic framework proposed in this paper.

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Appendix

Proof of proposition 1.

i) **Existence.** If it exists, $\hat{\delta}_{j,k}(\hat{T})$ is solution of $\frac{\lambda_j^*}{\lambda_k^*} = \mathrm{GWP}_{j,k}(\hat{T})$. Using (18) and (19) yields:

$$\hat{\delta}_{j,k}(\hat{T}) = \frac{\tau_j \tau_k (e^{-\tau_k \hat{T}} - e^{-\tau_j \hat{T}})}{\tau_j(1 - e^{-\tau_k \hat{T}}) - \tau_k(1 - e^{-\tau_j \hat{T}})} \tag{27}$$

Positivity. Let u and v be the numerator and the denominator in (27), respectively. Consider without loss of generality that $\tau_j > \tau_k$. We know that in this case $u > 0$. Consider the function $f(x) = \frac{\ln(x)}{1-\frac{1}{x}}$ and the change of variables $\ln(s) = \hat{T}\tau_j$ and $\ln(t) = \hat{T}\tau_k$. $f(x)$ is increasing for $x \geq 1$. Therefore, $\tau_j > \tau_k$ implies $s > t(> 1)$ and $f(s) > f(t)$. It leads to $\ln(s)(1 - 1/t) > \ln(t)(1 - 1/s)$ or $\hat{T}\tau_j(1 - e^{-\tau_k \hat{T}}) > \hat{T}\tau_k(1 - e^{-\tau_j \hat{T}})$. As $\hat{T} > 0$, we thus have $v > 0$. As $\hat{\delta}_{j,k}(\hat{T}) = \hat{\delta}_{k,j}(\hat{T})$, the proof for $\tau_j < \tau_k$ is straightforward.

Unicity. It is sufficient to show that $\hat{\delta}_{j,k}(T)$ is strictly monotonically decreasing with respect to T, that is $vu' - uv' < 0$ for all $\tau_j, \tau_k, T > 0$. Re-arranging and simplifying, it comes

$$vu' - uv' = \tau_j \tau_k (\tau_j - \tau_k) \left[\tau_j e^{-\tau_j T}(1 - e^{-\tau_k T}) - \tau_k e^{-\tau_k T}(1 - e^{-\tau_j T})\right] \tag{28}$$

Consider $g(x) = \frac{x \ln(x)}{1-\frac{1}{x}}$ and the same change of variables as above. $g(x)$ is decreasing with respect to x for $x \geq 1$. By the same token as above, $\tau_j > \tau_k$ implies $s > t(> 1)$ and $g(s) < g(t)$. Consequently, the term in square brackets in (28) is negative. QED.

ii) Using equation (27), we have: (a) if $v > 0$, then $\frac{\lambda_j^*}{\lambda_k^*} > \mathrm{GWP}_{j,k}(\hat{T}) \iff \delta > \hat{\delta}_{j,k}(\hat{T})$, and (b) if $v < 0$, then $\frac{\lambda_j^*}{\lambda_k^*} > \mathrm{GWP}_{j,k}(\hat{T}) \iff \delta < \hat{\delta}_{j,k}(\hat{T})$. From (i), we know that the sign of v is the same as that of u, which is the same as that of $(\tau_j - \tau_k)$. QED.

Proof of proposition 2.

i) The following notations are used: $\mathbf{A} = \mathbf{E}\mathbf{V}^{-1}\, {}^t\mathbf{E}$ and $\tilde{\mathbf{A}} = \frac{\mathbf{A}\boldsymbol{\gamma}\, {}^t\boldsymbol{\gamma}\mathbf{A}}{{}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}}$. Under (H4), we know that $\mathbf{V}$ is positive definite. $\mathbf{A}$ is thus positive semidefinite (definite if $rank(\mathbf{E}) = m$) and $\tilde{\mathbf{A}}$ is positive semidefinite.

Therefore, ${}^t\boldsymbol{\lambda}^*(\boldsymbol{\varepsilon}^{\text{BAU}} - \boldsymbol{\varepsilon}^*) = {}^t\boldsymbol{\lambda}^*\mathbf{A}\boldsymbol{\lambda}^* \geq 0$. Likewise, ${}^t\boldsymbol{\lambda}^*(\boldsymbol{\varepsilon}^{\text{BAU}} - \tilde{\boldsymbol{\varepsilon}}) = \frac{{}^t\boldsymbol{\gamma}\mathbf{A}\tilde{\boldsymbol{\mu}}}{{}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}}\,{}^t\boldsymbol{\lambda}^*\mathbf{A}\boldsymbol{\gamma}$. As $\boldsymbol{\lambda}^* = \tilde{\boldsymbol{\mu}}$ and $\mathbf{A}$ is symmetric under (H5), we have ${}^t\boldsymbol{\lambda}^*(\boldsymbol{\varepsilon}^{\text{BAU}} - \tilde{\boldsymbol{\varepsilon}}) = \frac{\left({}^t\boldsymbol{\gamma}\mathbf{A}\tilde{\boldsymbol{\mu}}\right)^2}{{}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}} \geq 0$. QED.

ii) We have ${}^t\boldsymbol{\lambda}^*(\tilde{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}^*) = {}^t\boldsymbol{\lambda}^*\mathbf{A}\boldsymbol{\lambda}^* - \frac{{}^t\boldsymbol{\gamma}\mathbf{A}\tilde{\boldsymbol{\mu}}}{{}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}}\,{}^t\boldsymbol{\lambda}^*\mathbf{A}\boldsymbol{\gamma}$. As $\boldsymbol{\lambda}^* = \tilde{\boldsymbol{\mu}}$ under (H1)-(H2) and $\mathbf{A}$ is symmetric and positive semi-definite, (ii) is proved as a direct application of the Cauchy-Schwarz inequality. QED.

iii) We have ${}^t\boldsymbol{\gamma}(\tilde{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}^*) = {}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\lambda}^* - \frac{{}^t\boldsymbol{\gamma}\mathbf{A}\tilde{\boldsymbol{\mu}}}{{}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}}\,{}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}$. As $\boldsymbol{\lambda}^* = \tilde{\boldsymbol{\mu}}$ under (H1)-(H2), (iii) is proved. QED.

Proof of proposition 3.

i) Let $\mathbf{M}$ (resp. $\tilde{\mathbf{M}}$) be the 4×4 matrix appearing in system (24) (resp. (26)). The computation of $|\mathbf{M}|$ yields $|\mathbf{M}| = \tau_1\tau_2(\delta+\tau_1)(\delta+\tau_2)\left[\beta\,{}^t\boldsymbol{\theta}\left(\boldsymbol{\Delta}_{[\tau]}^{-1}\mathbf{A}\boldsymbol{\Delta}_{[\delta+\tau]}^{-1}\right)\boldsymbol{\theta} + 1\right]$. As $\mathbf{A}$ is positive semidefinite, $|\mathbf{M}| > 0$. The same applies to the demonstration of $|\tilde{\mathbf{M}}| > 0$, with $\tilde{\mathbf{A}}$ replacing $\mathbf{A}$. QED.

ii) The computation of the eigenvalues of $\mathbf{M}$ yields:

$$\nu_{1/2} = \frac{1}{2}\left(\delta + \sqrt{\delta^2 + 2\left(Q \pm \sqrt{Q^2 - 4|\mathbf{M}|}\right)}\right),\ \nu_{3/4} = \frac{1}{2}\left(\delta - \sqrt{\delta^2 + 2\left(Q \pm \sqrt{Q^2 - 4|\mathbf{M}|}\right)}\right)$$

where $Q = \beta\,{}^t\boldsymbol{\theta}\mathbf{A}\boldsymbol{\theta} + \tau_1(\delta+\tau_1) + \tau_2(\delta+\tau_2) > 0$. ν_1 and ν_2 are both positive and are associated with the converging optimal paths. ν_3 and ν_4 are negative and are associated with the explosive optimal paths. The same applies to the calculation of the eigenvalues of $\tilde{\mathbf{M}}$, with $\tilde{\mathbf{A}}$ replacing $\mathbf{A}$. QED

Proof of proposition 4.

i) The block-inversion of $\mathbf{M}$ and computation of $\mathbf{M}^{-1}\begin{pmatrix} -\varepsilon^{BAU} \\ \mathbf{0} \end{pmatrix}$ yield:

$$\begin{pmatrix} \mathbf{z}^{*\infty} \\ \boldsymbol{\lambda}^{*\infty} \end{pmatrix} = \begin{pmatrix} \left(\boldsymbol{\Delta}_{[\tau]} + \beta\mathbf{A}\boldsymbol{\Delta}_{[\delta+\tau]}^{-1}\boldsymbol{\theta}\,{}^t\boldsymbol{\theta}\right)^{-1}\varepsilon^{BAU} \\ \boldsymbol{\Delta}_{[\delta+\tau]}^{-1}\mathbf{A}\left(\boldsymbol{\Delta}_{[\tau]} + \beta\mathbf{A}\boldsymbol{\Delta}_{[\delta+\tau]}^{-1}\boldsymbol{\theta}\,{}^t\boldsymbol{\theta}\right)^{-1}\varepsilon^{BAU} \end{pmatrix} \tag{29}$$

By using equation (29) for $\boldsymbol{\lambda}^{*\infty}$ and re-arranging (in the case $n = 2$), it comes:

$$\lambda_1^{*\infty} = \frac{\theta_1\tau_2\varepsilon_1^{BAU} + \theta_2\tau_1\varepsilon_2^{BAU}}{|\mathbf{M}|}\beta(\delta+\tau_2)\theta_1 \quad (30)$$

$$\lambda_2^{*\infty} = \frac{\theta_1\tau_2\varepsilon_1^{BAU} + \theta_2\tau_1\varepsilon_2^{BAU}}{|\mathbf{M}|}\beta(\delta+\tau_1)\theta_2 \quad (31)$$

The computation of the shadow price ratio follows. The same reasoning applies to $\frac{\tilde{\mu}_2^{\infty}}{\tilde{\mu}_1^{\infty}}$, with $\tilde{\mathbf{M}}$ replacing $\mathbf{M}$. QED.

ii) From equations (30)–(31), one easily sees that $\frac{\tilde{\mu}_1^{\infty}-\lambda_1^{*\infty}}{\lambda_1^{*\infty}} = \frac{\tilde{\mu}_2^{\infty}-\lambda_2^{*\infty}}{\lambda_2^{*\infty}} = \left(|\mathbf{M}| - |\tilde{\mathbf{M}}|\right)/|\tilde{\mathbf{M}}|$. The sign of $\tilde{\mu}_j^{\infty} - \lambda_j^{*\infty}$ $(j = 1, 2)$ is thus the same as that of $|\mathbf{M}| - |\tilde{\mathbf{M}}|$.

$$|\mathbf{M}| - |\tilde{\mathbf{M}}| = \frac{\beta\gamma_1^2\theta_1^2\tau_2(\delta+\tau_2)}{{}^t\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}}|\mathbf{A}|\left(\frac{\gamma_2}{\gamma_1} - \frac{\theta_2(\delta+\tau_1)}{\theta_1(\delta+\tau_2)}\right)\left(\frac{\gamma_2}{\gamma_1} - \frac{\theta_2\tau_1}{\theta_1\tau_2}\right) \quad (32)$$

As $\mathbf{A}$ is positive semidefinite (definite if rank($\mathbf{E}$)$= n$), part (ii) is proved. QED.

iii) The proof simply consists in introducing $\frac{\gamma_2}{\gamma_1} = \frac{\theta_2(\delta+\tau_1)}{\theta_1(\delta+\tau_2)}$ in equation (29), rearranging, which leads to $\mathbf{z}^{*\infty} = \tilde{\mathbf{z}}^{\infty}$. QED.